Th 1. m^* is countably subadditive.

Warning: if $\#(I)$ is not countable $A = \bigcup_{i \in I} A_i$ then not true that $m^*(A) \leq \sum_{i \in I} m^*(A_i)$.

Pf. Let $\varepsilon > 0$. Consider $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$, and assume wlg that $m^*(A_i) < +\infty$. Then, by definition, $\exists$ COIC $\{I_n^i : n \in \mathbb{N}\}$ of A_i such that

$$m^*(A_i) + \frac{\varepsilon}{2^i} > \sum_{n=1}^{\infty} l(I_n^i).$$

Do this for all $i \in \mathbb{N}$ and we have a COIC $\{I_n^i : i, n \in \mathbb{N}\}$ of $A := \bigcup_{i \in \mathbb{N}} A_i$

and so

$$m^*(A) \leq \sum_{i=1}^{n} \sum_{n=1}^{\infty} l(I_n^i) \leq \sum_{i=1}^{\infty} m^*(A_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}$$
$$= \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon$$

Consequently ....

Lemma. Suppose $\mathcal{C}$ is a finite open interval cover of $[a,b]$ (say $\#(\mathcal{C}) = n \in \mathbb{N}$). Then the length of $[a,b]$ is $\leq$ total length of $\mathcal{C}$.

(I) MI over $n = \#(\mathcal{C})$ ← please complete this method of Proof.

Proof (II) (i) Take $(a_1, b_1) \in \mathcal{C}$ s.t. $b \in (a_1, b_1)$.
May assume w.l.g. that $b_1 < b$ (otherwise $b - a \leq b_1 - a_1$).

(ii) Take $(a_2, b_2) \in \mathcal{C}$ s.t. $b_1 \in (a_2, b_2)$.
May assume w.l.g that $b_2 < b$.

(iii) Take $(a_3, b_3) \in \mathcal{C}$ s.t. $b_2 \in (a_3, b_3)$.

Since $\#(\mathcal{C}) = n \in \mathbb{N}$, this process must end after N-many steps for some $N \leq n$, i.e.

$$b \leq b_N.$$

while

$$b_i < b \quad \text{and} \quad b_i \in (a_{i+1}, b_{i+1}) \ \forall\, i = 1, \dots, N-1.$$

Note then

$$\begin{aligned}
b - a &\leq b_N - a_1 \\
&\leq (b_N - a_N) + (b_{N-1} - a_1) \qquad (\because b_{N-1} \in (a_N, b_N)) \\
&\leq (b_N - a_N) + (b_{N-1} - a_{N-1}) + (b_{N-2} - a_1) \\
&\leq \cdots \qquad (\because b_{N-2} \in (a_{N-1}, b_{N-1})) \\
&\leq (b_N - a_N) + \cdots + (b_2 - a_1) \\
&\leq \cdots\cdots + (b_2 - a_2) + (b_1 - a_1)
\end{aligned}$$

Thm $m^*([a,b]) = $ length of $[a,b]$. $(*)$

Pf. $m^*([a,b]) \leq \ell((a-\varepsilon/2, b+\varepsilon/2)) = b-a+\varepsilon \ \forall \varepsilon > 0$

so LHS $\leq$ RHS of $(*)$

Let $\mathcal{C}$ be an open interval cover of $[a,b]$. By Heine-Borel it has a finite subcover $\mathcal{C}_0$. Then

total length of $\mathcal{C}$ $\geq$ total length of $\mathcal{C}_0 \geq \ell([a,b])$

by the preceding Lemma

and it follows from the def of $m^*([a,b])$ that

$$m^*([a,b]) \geq b-a$$

Th3. $(a,+\infty)$ is measurable.

Pf. Let $m^*(A) < +\infty$ and

$$A_1 = (a,+\infty) \cap A$$
$$A_2 = (-\infty, a) \cap A \quad (\text{of the same outer mea. of } (-\infty, a] \cap A)$$

We need only show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2) \qquad (\#)$$

Let $\{I_n : n \in \mathbb{N}\}$ be a COIC of A. Let

$$I_n' = \quad \cap I_n \qquad \forall n \in \mathbb{N}.$$
$$I_n'' = \quad \cap I_n$$

Then they are resp. COIC of A_1, A_2 and so

$$\sum_{n=1}^{\infty} l(I_n') \geqslant m^*(A_1)$$

$$\sum_{n=1}^{\infty} l(I_n'') \geqslant m^*(A_2);$$

Consequently, as $l(I_n) = l(I_n') + l(I_n'')$

$$\sum_{n=1}^{\infty} l(I_n) \geqslant m^*(A_1) + m^*(A_2).$$

It follows that

$$m^*(A) \geqslant m^*(A_1) + m^*(A_2)$$

by def of $m^*(A)$.